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The Entropy Formula for SRB-Measures of Lattice Dynamical Systems

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We give a detailed proof of the entropy formula for SRB-measures of coupled hyperbolic attractors over integer lattices. We show that the topological pressure for the potential function of the SRB-measure is zero.

KEY WORDS: Entropy formula; SRB measure; coupled map lattice; topological pressure.

1. INTRODUCTION

We proved in ref. 3 that the thermodynamic limit of Sinai–Ruelle–Bowen measures for coupled hyperbolic maps over finite volumes of an integer lattice exists as the volume tends to infinity. The limiting measure, also called an SRB-measure, is an equilibrium state satisfying the variational principle of statistical mechanics for a Hölder continuous function φ_0 on an infinite-dimensional hyperbolic attractor. The measure is invariant and exponentially mixing with respect to both spatial and temporal translations. The formula for computing the potential function φ_0 is explicitly given. In this note, we give a detailed proof of a result in ref. 3 stating that the topological pressure for this potential function is zero with respect to the group actions induced by both spatial and temporal translations. Thus, the entropy formula holds for the SRB-measure for the coupled hyperbolic map lattice. This result further justifies the name of the measure since the topological pressure being zero is one of the characteristics of an SRB-measure on hyperbolic attractors of finite dimension. It also paves the way

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for estimating the measure theoretical (space-time) entropy in terms of hyperbolicity and coupling strength of coupled hyperbolic map lattices.

The proof is a rather straightforward computation using the definition of topological pressures for continuous functions on a compact metric space with respect to a $\mathbb{Z}^d$-action induced by d interchangeable homeomorphisms. First, we briefly describe the infinite-dimensional system: weakly coupled identical systems with a uniformly hyperbolic attractor and the results concerning its SRB-measure. We then use the conjugacy between hyperbolic systems and their symbolic representations and special properties of the potential function φ_0 to show that its topological pressure is zero.

2. SRB-MEASURES FOR COUPLED MAP LATTICES

Let M be a smooth compact Riemannian manifold and f be a C^r-diffeomorphism of M, $r > 1$. We assume that f possesses a hyperbolic attractor Λ, i.e., f is uniformly hyperbolic on Λ and there exists an open neighborhood $U \supset \Lambda$ such that $f(U) \subset U$ and $\bigcap_{k=1}^{\infty} f^k(U) = \Lambda$.

A Sinai–Ruelle–Bowen measure μ for f on the hyperbolic attractor is described by the following limit:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k(x)) = \int g \, d\mu$$

where the equality holds for any continuous function g on M and for almost all $x \in U$ with respect to the Lebesgue measure.[1] When, f is topologically transitive on Λ, this measure μ is also unique and is the unique equilibrium state that satisfies the entropy formula

$$h_\mu(f) = -\int \varphi_f(x) \, d\mu$$

where $h_\mu(f)$ is the measure theoretical entropy and $\varphi_f(x) = -\log J^u f(x)$ ($J^u f(x)$ is the Jacobian of the restriction of f along the unstable manifold at x).

Since we shall consider SRB-measures for direct product spaces of (M, f), we further assume that f is topologically mixing on Λ. This assumption will ensure uniqueness of SRB-measures since product spaces of topologically mixing systems are still topologically mixing and topological transitivity is not preserved by the direct product.

The direct product of identical copies of M over a d-dimensional integer lattice $\mathcal{M} = \bigotimes_{i \in \mathbb{Z}^d} M_i$ is an infinite-dimensional Banach manifold with a Finsler metric induced by the Riemannian metric on M.

The distance on $\mathscr{M}$ induced by the Finsler metric is

$$\rho(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}d} d(x_i, y_i)$$

where $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are two points in $\mathscr{M}$ and d is the Riemannian distance on M.

The *direct product* map on $\mathscr{M}$ defined by $F = \bigotimes_{i \in \mathbb{Z}^d} f_i$ possesses an infinite-dimensional hyperbolic attractor $\Lambda_F = \bigotimes_{i \in \mathbb{Z}^d} \Lambda_i$, where f_i and Λ_i are copies of f and Λ, respectively.

We recall the definitions of some objects discussed in ref. 3.

Let S denote the spatial translations on $\mathscr{M}$ induced by the translations on the integer lattice $\mathbb{Z}^d$, i.e., for any $k \in \mathbb{Z}^d$ and $\bar{x} = (x_i) \in \mathscr{M}$, $S^k(\bar{x}) = (x_{i+k})$. Let the map G be a C^2-perturbation of the identity map on $\mathscr{M}$. G is said to be spatially translation invariant if $G \circ S = S \circ G$. It is said to have *short range* property if G, written in the form $G = (G_i)_{i \in \mathbb{Z}^d}$, where G_i: $\mathscr{M} \to M_i$, has the following property: there exist a *decay constant* θ, $0 < \theta < 1$ and a constant $C > 0$ such that for any fixed $k \in \mathbb{Z}^d$ and any points $\bar{x} = (x_j)$, $\bar{y} = (y_j) \in \mathscr{M}$ with $x_j = y_j$ for all $j \in \mathbb{Z}^d$, $j \neq k$,

$$d(G_i(\bar{x}), G_i(\bar{y})) \leqslant C\theta^{|i-k|} d(x_k, y_k)$$

Define $\Phi = F \circ G$ (or equivalently, $\Phi = G \circ F$, since F is also a diffeomorphism). The map Φ is a perturbation of F. The infinite-dimensional dynamical system $(\mathscr{M}, (\Phi, S))$ is called a *coupled map lattice*. If $G = id$, the lattice is called *uncoupled*. When G is spatially translation invariant, Φ satisfies the same property and the pair (Φ, S) generates a $\mathbb{Z}^{d+1}$-action on $\mathscr{M}$.

In order to study invariant measures of coupled map lattices, a family of metrics compatible with the Tychonov compact topology (the direct product topology) on $\mathscr{M}$ is introduced. The metric ρ_q, $0 < q < 1$, is defined by

$$\rho_q(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}^d} q^{|i|} d(x_i, y_i)$$

where $|i| = |i_1| + |i_2| + \cdots + |i_d|$, $i = (i_1, i_2, ..., i_d) \in \mathbb{Z}^d$.

We state the main results in refs. 2 and 3 on the existence of SRB-measures for Φ and the properties of this measure.

(1) For any $\varepsilon > 0$ there exists $0 < \delta < \delta_0$ such that, if $\mathrm{dist}_{C^1}(\Phi, F) \leqslant \delta$, then there is a unique homeomorphism h: $\Lambda_F \to \mathscr{M}$ satisfying $\Phi \circ h = h \circ F|_{\Lambda_F}$ with $\mathrm{dist}_{C^0}(h, id) \leqslant \varepsilon$. In particular, the set $\Lambda_\Phi = h(\Lambda_F)$ is a

topologically mixing hyperbolic attractor for the map Φ. The conjugating map h is spatial translation invariant whenever G is.

(2) For any $0<\theta<1$ there exists $\delta>0$ such that if G is a C^2-spatial translation invariant short range map with a decay constant θ and $\mathrm{dist}_{C^1}(G, id) \leqslant \delta$, then the conjugacy map h is Hölder continuous with respect to the metric ρ_q, $0<q<1$. Moreover, $h=(h_i(\bar{x}))_{i\in\mathbb{Z}^d}$ satisfies the following property:

$$d(h_0(\bar{x}), h_0(\bar{y})) \leqslant C(\delta)\, d^{\alpha}(x_k, y_k)$$

for every $k \neq 0$ and any $\bar{x}, \bar{y} \in \mathcal{M}$ with $x_i = y_i$, $i \in \mathbb{Z}^d$, $i \neq k$, where α, $0<\alpha<1$, and $C(\delta)>0$ are constants. Furthermore, $C(\delta) \to 0$ as $\delta \to 0$.

Fix a point $\bar{x}^* \in \Lambda_\Phi$, and a finite volume $V \subset \mathbb{Z}^d$, the map $\Phi_V \colon x_V \to \Phi_V(x_V)$ on $M_V = \bigotimes_{i\in V} M_i$ is defined coordinatewisely by

$$(\Phi_V(x_V))_i = (\Phi((x_V, x^*|_{\hat{V}}))_i, \qquad i \in V$$

where the point $(x_V, x^*|_{\hat{V}})$ denotes an element in $\mathcal{M}$ whose restrictions to V and its complement $\hat{V}$ are x_V and $x^*|_{\hat{V}}$, respectively.

The map Φ_V is a diffeomorphism of M_V when the perturbation G is sufficiently close to identity and it is C^1-closed to the diffeomorphism $F_V = \bigotimes_{i\in V} f$. By the structural stability theorem Φ_V possesses a hyperbolic attractor Λ_{Φ_V} since F_V has a hyperbolic attractor $\Lambda_{F_V} = \bigotimes_{i\in V} \Lambda$. There exists a conjugating homeomorphism $h_V \colon \Lambda_{F_V} \to \Lambda_{\Phi_V}$, $\Phi_V \circ h_V = h_V \circ F_V$.

The maps Φ_V and h_V provide finite-dimensional approximations for the infinite-dimensional maps Φ and h, respectively.

(3) Let μ_V be the SRB-measure on the hyperbolic attractor Λ_{Φ_V} for the map Φ_V. Then, the measure μ_V weakly converges to a measure μ on Λ_Φ. The measure μ is invariant and exponentially mixing under Φ and spatial translations S. It also satisfies the variational principle:

$$P_\tau(\varphi_0) = h_\mu(\tau) + \int \varphi_0 \, d\mu$$

where τ denotes the $\mathbb{Z}^{d+1}$ action on Λ_Φ induced by Φ and S, $P_\tau(\varphi_0)$ is the topological pressure for the potential function φ_0, and $h_\mu(\tau)$ is the measure theoretical entropy of μ with respect to τ.

(4) The construction of the potential function φ_0 can be described in the following way. By assumption, the map $\Phi = F \circ G$ is C^1-close to F and the map G has short range property, under an appropriately chosen local coordinate system, the restriction of the derivative operator of Φ_V to the

unstable space $E^u_{\Phi_V}$ at point $h_V(x_V)$ has the following matrix representation:

$$D\Phi|_{E^u_{\Phi_V}(h_V(x_V))} = (D^u f(x_i))(I + A_V(x_V))$$

where $A_V(x_V) = (\mathbf{a}_{ij}(x_V))$ is a $|V| \times |V|$ matrix with submatrices $\mathbf{a}_{ij}(x_V)$ as its entries, $(D^u f(x_i))$ is a diagonal matrix with $D^u f(x_i)$, $i \in V$ (the matrix representation of Df restricted to the unstable space) on its main diagonal, and $|V|$ is the cardinality of V. Note that $x_V \in \Delta_{F_V}$. The norms of submatrices $\mathbf{a}_{ij}(x_V)$ are small and go to zero exponentially fast as $|i-j| \to \infty$. The entries $\mathbf{a}_{ij}(x_V)$ are also Hölder continuous with respect to the metric ρ_q. The determinant of $(I + A_V)$ is then expanded in the following way.

$$\det(I + A_V) = \exp(\operatorname{trace}(\ln(I + A_V)) = \exp\left(-\sum_{i \in V} w_{Vi}\right)$$

where

$$w_{Vi}(x_V) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{trace}(\mathbf{a}^n_{ii}(x_V))$$

and $\mathbf{a}^n_{ii}(x_V)$ are submatrices on the main diagonal of $(A_V)^n$.

The functions $w_{Vi}(x_V)$ have the following properties. There exist constants $\varepsilon_0 > 0$, $\beta > 0$ such that

$$|w_{Vi}(x_V) - w_{V'i}(y_{V'})| \leqslant \varepsilon_0 e^{-\beta d(i, V)} \tag{1}$$

where $V \subset V'$, $x_V = y_{V'}|_V$, and $d(i, V)$ denotes the distance from the lattice site i to the boundary of V. The estimation (1) implies that the limit $\psi_i(\bar{x}) = \lim_{V \to \mathbb{Z}^d} w_{Vi}(x_V)$ exists for each $i \in \mathbb{Z}^d$. This limit is also translation invariant in the following sense. Let $\psi(\bar{x}) = \lim_{V \to \mathbb{Z}^d} w_{V0}(x_V)$. Then, $\psi_i(\bar{x}) = \psi(S^i \bar{x})$. Moreover, $\psi(\bar{x})$ is Hölder continuous with respect to the metric ρ_q with a Hölder constant going to zero as the C^1-distance between Φ and F tends to zero.

The potential function φ_0 for the SRB-measure for the coupled map lattice (Φ, S) composed with the conjugating map h is

$$\varphi_0(h(\bar{x})) = -\log J^u f(x_0) + \psi(\bar{x}) \tag{2}$$

The potential function is expressed slightly differently from that in ref. 3 since we have a hyperbolic attractor instead of an Anosov system and φ_0 is described by a composition for the convenience of later computation.

3. COMPUTING THE TOPOLOGICAL PRESSURE

In this section we prove that the topological pressure of the potential function φ_0 with respect to the $\mathbb{Z}^{d+1}$-action induced by the coupled map lattice (Φ, S) is zero on the hyperbolic attractor $\Lambda_\Phi = h(\Lambda_F) : P_\tau(\varphi_0) = 0$. Therefore, the entropy formula $h_\mu(\tau) = -\int \varphi_0 \, d\mu$ holds for the measure μ. We first recall the definition of the topological pressure.[6]

3.1. Topological Pressure

Let Ω be a compact metric space and τ a $\mathbb{Z}^{d+1}$-action on Ω induced by $d+1$ commuting homeomorphisms, $d \geqslant 0$. For any two covers of Ω $\mathscr{U} = \{U_i\}$ and $\mathscr{B} = \{B_i\}$, $\mathscr{U} \vee \mathscr{B}$ denotes the cover of Ω consisting of all sets of the form $B_i \cap U_j$. For a finite volume $V \subset \mathbb{Z}^{d+1}$ define

$$\mathscr{U}^V = \bigvee_{i \in V} \tau^{-i} \mathscr{U}$$

Let $\mathscr{U}$ be any cover of Ω, φ a continuous function on Ω, and V a finite subset of $\mathbb{Z}^{d+1}$. The *partition function* is defined by

$$Z_V(\varphi, \mathscr{U}) = \min_{\{B_j\}} \left\{ \sum_j \exp \left[\sup_{x \in B_j} \sum_{i \in V} \varphi(\tau^i x) \right] \right\} \tag{3}$$

where the minimum is taken over all subcovers $\{B_j\}$ of $\mathscr{U}^V$. Because of the subadditivity of the partition function, the following limit exists and is call the topological pressure with respect to the cover $\mathscr{U}$.

$$P_\tau(\varphi, \mathscr{U}) = \lim_{V \uparrow \infty} \frac{1}{|V|} \log Z_{V(a)}(\varphi, \mathscr{U})$$

where $V \uparrow \infty$ means V approaches $\mathbb{Z}^{d+1}$ in the sense of Van Hove. When $\mathscr{U}$ is an open cover, the quantity

$$P_\tau(\varphi) = \lim_{\operatorname{diam} \mathscr{U} \to 0} P_\tau(\varphi, \mathscr{U}) = \sup_{\mathscr{U}} P_\tau(\varphi, \mathscr{U})$$

is called the *topological pressure* of φ with respect to τ.

It is easy to see that for a fixed volume V_0

$$P_\tau(\varphi, \mathscr{U}) = P_\tau(\varphi, \mathscr{U}^{V_0})$$

When τ is expansive and diam $\mathscr{U}$ is smaller than the expansive constant, we have $P_\tau(\varphi) = P_\tau(\varphi, \mathscr{U})$.

Before we proceed to the actual computation, we first state the strategy:

First of all, we project every object onto the hyperbolic attractor Δ_F for F. The metrics we are using now is the metric ρ_q. Since the conjugating map h is Hölder continuous, it is easy to see that $P_\tau(\varphi) = P_\tau(\varphi(h(\bar{x}))$. It is much easier to compute $P_\tau(\varphi(h(\bar{x}))$ since the $\mathbb{Z}^{d+1}$-action τ induced by (F, S) is now acting on the direct product space $\Delta_F = \bigotimes_{i \in \mathbb{Z}^d} \Lambda_i$.

Next, we use the Markov partition of Λ for the map f to obtain a natural symbolic representation of the system $(\Delta_F, (F, S))$. Using a result from an earlier paper,[2] we can compute the topological pressure on the symbolic space where the discrete topology is an advantage.

In the actually computation of the partition function for the potential function $\varphi_0(h(\bar{x}))$ with respect to the $\mathbb{Z}^{d+1}$-action τ, we compare it with the partition function of the potential function $-\log J^u\Phi_V(h_V(x_V))$ for the SRB-measure of Φ_V projected onto the hyperbolic attractor Δ_{F_V}. Note that this partition function is computed with respect to the $\mathbb{Z}$-action generated by F_V. Using the fact that the pressure is zero for $-\log J^u\Phi_V(h_V(x_V))$. We can prove that the pressure for φ_0 is also zero.

3.2. Markov Partition and Symbolic Representation

Since the set Λ is a hyperbolic attractor for the diffeomorphism f, for any $\varepsilon > 0$, there exists a Markov partition of Λ into *proper rectangles*:

$$\Lambda = \bigcup_{i=1}^{p} R_i$$

where $\{R_i\}$ satisfy the following properties.[1, 5]

(1) diam $\mathscr{R} = \max_i \operatorname{diam}(R_i) < \varepsilon$.

(2) For each i, R_i is *proper*: $\overline{\operatorname{int} R_i} = R_i$.

(3) For any two points $x, y \in R_i$, there is a unique point in the intersection of the local stable manifold at x, $W^s_\varepsilon(x)$ and the local unstable manifold at y, $W^u_\varepsilon(y)$. This point denoted by $[x, y]$ is also in R_i: $[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \in R_i$, i.e. R_i is a *rectangle*.

(4) $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ when $i \neq j$.

When ε is sufficiently small, for each $x \in R_i$, we can also assume that the intersection $W^s_\varepsilon(x) \cap R_i$ (denoted by $W^s(x, R_i)$) is proper and its boundary $\partial W^s(x, R_i)$ is in ∂R_i, the boundary of R_i. Similarly, we define $W^u(x, R_i)$. Note that $W^u(x, R_i)$ is a submanifold with boundary, but

$W^s(x, R_i)$ is, in general, not a submanifold. We use relative topologies on both sets. These sets are invariant under f:

(5) $f(W^s(x, R_i)) \subset W^s(f(x), R_j), \qquad W^u(x, R_i) \subset f^{-1}(W^u(f(x), R_j)),$
when $x \in \text{int}\, R_i$ and $f(x) \in \text{int}\, R_j$.

Define a $p \times p$ matrix $A = (a_{ij})$ by the following rule:

$$a_{ij} = \begin{cases} 1, & \text{if } \ \text{int}\, R_i \cap f^{-1}(\text{int}\, R_j) \neq \emptyset \\ 0, & \text{if } \ \text{int}\, R_i \cap f^{-1}(\text{int}\, R_j) = \emptyset \end{cases}$$

Since we assumed that the map f is topologically mixing, the matrix A is aperiodic, i.e., there exists a constant n_0 depending on A such that all entries of A^{n_0} are positive. Let Σ_A be the subshift of finite type determined by the matrix A. For each element of Σ_A, $\xi = (\xi(j))_{j \in \mathbb{Z}}$, define a map $\pi: \Sigma_A \to \Lambda$ by

$$\pi(\xi) = \bigcap_{j \in \mathbb{Z}} f^{-j}(R_{\xi(j)})$$

The map π acts as a semi-conjugacy between Λ and its symbolic representation Σ_A. This semi-conjugacy can be naturally extended to hyperbolic attractors $\Lambda_{F_V} = \bigotimes_{i \in V} \Lambda$ and Λ_F.

$$\pi_V: \bigotimes_{i \in V} \Sigma_A \to \Lambda_{F_V}$$

$$\bar{\pi}: \bigotimes_{i \in \mathbb{Z}^d} \Sigma_A \to \Lambda_F$$

For any element $\bar{\xi} = (\xi_i)_{i \in \mathbb{Z}^d} \in \bigotimes_{i \in \mathbb{Z}^d} \Sigma_A$, $\bar{\pi}(\bar{\xi}) = (\pi \xi_i)_{i \in \mathbb{Z}^d}$.

We introduce the corresponding metric ρ_q, $0 < q < 1$ on the symbolic space $\bigotimes_{i \in \mathbb{Z}^d} \Sigma_A$.

$$\rho_q(\bar{\xi}, \bar{\eta}) = \sup_{i \in \mathbb{Z}^d,\, j \in \mathbb{Z}} q^{|i| + |j|} d(\xi_i(j), \eta_i(j))$$

where $d(\cdot, \cdot)$ denotes the discrete distance on the set $\{1, 2, ..., p\}$. The definition of the metric ρ_q is valid for Σ_A and $\otimes_{i \in V} \Sigma_A$, $V \subset \mathbb{Z}^d$. For Σ_A, we can simply choose $i = 0$. The map induced by the shift on Σ_A will be denoted by σ_t on all these symbolic spaces. The maps induced by translations on $\mathbb{Z}^d$ are denoted by σ_s. We have $f \circ \pi = \pi \circ \sigma_t$, $F_V \circ \pi_V = \pi_V \circ \sigma_t$, $F \circ \bar{\pi} = \bar{\pi} \circ \sigma_t$, and $S \circ \bar{\pi} = \bar{\pi} \circ \sigma_s$.

The next two propositions enable us to pass the computation of topological pressures for functions on hyperbolic attractors onto their symbolic representations.

Proposition 1. The topological pressure for any Hölder continuous function φ on Δ_{F_V} with respect to the map F_V is equal to the topological pressure for $\varphi \circ \pi_V$ on $\bigotimes_{i \in V} \Sigma_A$ with respect to the shift σ_t.

$$P_{F_V}(\varphi) = P_{\sigma_t}(\varphi \circ \pi_V)$$

In particular, for $\varphi_V = -\log J^u \Phi_V$, the potential function for the SRB-measure on Δ_{Φ_V} for Φ_V, we have

$$P_{\Phi_V}(\varphi_V) = P_{F_V}(\varphi_V \circ h_V) = P_{\sigma_t}(\varphi_V \circ h_V \circ \pi_V) = 0 \tag{4}$$

Proposition 2. For the map f and any constant α, $0 < \alpha < 1$, there exists a constant $c_0 > 0$ such that if ψ is any function on Δ_F satisfying the (small Hölder constant) condition

$$|\psi(\bar{x}) - \psi(\bar{y})| \leqslant c_0 \rho_q^\alpha(\bar{x}, \bar{y})$$

then the topological pressure for the function $\varphi(\bar{x}) = -\log J^u f(x_0) + \psi(\bar{x})$ on Δ_F under the $\mathbb{Z}^{d+1}$-action τ induced by the maps F and S is equal to the topological pressure for the function $\varphi \circ \bar{\pi}$ on the symbolic space under the $\mathbb{Z}^{d+1}$-action (denoted by τ^*) induced by the maps σ_t and σ_s. In particular, for the function φ_0, the potential function for the SRB-measure for (Φ, S) on Δ_Φ, we have

$$P_\tau(\varphi_0) = P_{\tau^*}(\varphi_0 \circ h \circ \bar{\pi}) \tag{5}$$

Proposition 1 is standard[1] since the hyperbolic attractor is finite dimensional. For the detail of the proof of Proposition 2 see Theorems 6 and 7 of ref. 2.

3.3. Zero Topological Pressure

In this section, we prove that

$$P_\tau(\varphi_0) = P_{\tau^*}(\varphi_0 \circ h \circ \bar{\pi}) = 0$$

using Eq. (4). The advantage of computing topological pressure on the symbolic space is that the partition function takes a simpler form because of the discrete topology on the set of p symbols.

Theorem 1. Let φ_0 be the potential function for the SRB-measure μ defined in (2). Then,

$$P_\tau(\varphi_0) = 0$$

Therefore, the entropy formula holds.

$$h_\tau(\mu) = -\int \varphi_0 \, d\mu$$

Proof. We shall show that $P_{\tau^*}(\varphi_0 \circ h \circ \bar{\pi}) = 0$. For simplicity, we assume $d = 1$ and V is an interval $[-m, m]$. We denote Φ_V by Φ_m, φ_V by φ_m, and π_V by π_m, etc. For $d > 1$, the proof is the same. For every fixed m, n, we define a set Ω_{mn} as follows.

$$\begin{aligned}\Omega_{mn} = \{(\xi_j(j)),\ -m < i < m,\ -n < j < n \mid &\text{there exists an element} \\ &\bar{\eta} = (\eta_i(j))_{i,\, j \in \mathbb{Z}} \in \bigotimes_{i \in \mathbb{Z}} \Sigma_A \text{ such that} \\ &\eta_i(j) = \xi_i(j),\ -m < i < m,\ -n < j < n\}\end{aligned}$$

The set Ω_{mn} can be thought of as the restriction of the symbolic space $\bigotimes_{i \in \mathbb{Z}} \Sigma_A$ (or $\bigotimes_{i=-m}^{m} \Sigma_A$) over the region $[-m, m] \otimes [-n, n]$.

One can easily verify that the topological pressures take the simpler form below (see also, ref. 6).

$$P_{\sigma_t}(\varphi_m \circ \pi_m) = \lim_{n \to \infty} \frac{1}{2n} \log \sum_{\xi \in \Omega_{mn}} \exp \sum_{j=-n}^{n} \varphi_m(\sigma_t^i(\xi_m^*)))$$

where the element $\xi_m^* \in \bigotimes_{i=-m}^{m} \Sigma_A$ is so chosen that its restriction to $[-m, m] \otimes [-n, n]$ is $\xi \in \Omega_{mn}$.

$$P_{\tau^*}(\varphi_0 \circ h \circ \bar{\pi}) = \lim_{n, m \to \infty} \frac{1}{4mn} \log \sum_{\xi \in \Omega_{mn}} \exp \sum_{i=-m,\, j=-n}^{m, n} \varphi_0(h(\bar{\pi}(\sigma_t^j \sigma_s^i(\bar{\xi}^*))))$$

where the element $\bar{\xi}^* \in \bigotimes_{i \in \mathbb{Z}^d} \Sigma_A$ is so chosen that its restriction to $[-m, m] \otimes [-n, n]$ is $\xi \in \Omega_{mn}$.

Let

$$\begin{aligned} a_{mn} &= \frac{1}{2n} \log \sum_{\xi \in \Omega_{mn}} \exp \sum_{j=-n}^{n} \varphi_m(\pi_m(\sigma_t^i(\xi_m^*))) \\ &= \frac{1}{2n} \log \sum_{\xi \in \Omega_{mn}} \exp \sum_{j=-n}^{n} -\log J^u \Phi_m(h_m(F_m^i(\pi_m(\xi_m^*)))) \end{aligned}$$

Then, we have $\lim_{n \to \infty} a_{mn} = 0$.

Let

$$b_{mn} = \frac{1}{4mn} \log \sum_{\xi \in \Omega_{mn}} \exp \sum_{i=-m,\, j=-n}^{m,\, n} \varphi_0(h(\bar{\pi}(\sigma_t^j \sigma_s^i(\bar{\xi}^*))))$$

$$= \frac{1}{4mn} \log \sum_{\xi \in \Omega_{mn}} \exp \sum_{i=-m,\, j=-n}^{m,\, n} \varphi_0(h(F^j S^i(\bar{\pi}(\bar{\xi}^*))))$$

Then, $P_\tau(\varphi_0) = \lim_{m,\, n \to \infty} b_{mn}$.

Since $\lim_{n \to \infty} a_{mn} = 0$ for each m, we can find a sequence $\{n(m)\}$ such that

$$\lim_{m \to \infty} \frac{1}{2m} a_{mn(m)} = 0$$

Note that the volume $V_{mn} = [-m, m] \otimes [-n(m), n(m)] \to \mathbb{Z}^2$ in the sense of van Hove. Therefore, $P_\tau(\varphi_0) = \lim_{m \to \infty} b_{mn(m)}$. So, we need only to show that

$$\lim_{m \to \infty} b_{mn(m)} - \frac{1}{2m} a_{mn(m)} = 0$$

We now use the decomposition of the Jacobian

$$J^u \Phi_m(h_m(\bar{x})) = \prod_{i=-m}^{m} J^u f_i(x_i) \exp\left(-\sum_{i=-m}^{m} w_{mi}(\bar{x})\right)$$

and the definition of the function φ_0:

$$\varphi_0(h(\bar{x})) = \lim_{m \to \infty} w_{m0}(\bar{x}) - \log J^u f(x_0)$$

$$= \psi(\bar{x}) - \log J^u f(x_0)$$

$$\lim_{m \to \infty} w_{mi}(\bar{x}) = \psi(S^i \bar{x})$$

$$b_{mn} - \frac{1}{2m} a_{mn} = \frac{1}{4mn} \log \frac{\sum_{\xi \in \Omega_{mn}} \exp \sum_{i,\, j=-m,\, -n}^{m,\, n} \varphi_0(h(\bar{\pi}(\sigma_t^j \sigma_s^i(\bar{\xi}^*))))}{\sum_{\xi \in \Omega_{mn}} \exp \sum_{j=-n}^{n} -\log J^u \Phi_m(h_m(\pi_m(\sigma_t^j(\xi_m^*))))} \tag{6}$$

Since there are same number of terms in the numerator and the denominator in the logarithm in (6), we simply need to estimate the following expression.

$$D=\left|\sum_{i,\, j=-m,\, -n}^{m,\, n} \varphi_0(j(\bar{\pi}(\sigma_t^j\sigma_s^i(\bar{\xi}^*))))\right.$$
$$\left.-\sum_{j=-n}^{n} -\log J^u\Phi_m(h_m(\pi_m(\sigma_t^j(\xi_m^*))))\right|$$

Plug in the formulas for both $\varphi_0(h)$ and $J^u\Phi_m(h_m)$. Denote $\bar{\pi}(\bar{\xi}^*)=\bar{x}$ and $\pi_m(\xi_m^*)=y$. We choose ξ_m^* and $\bar{\xi}^*$ so that ξ_m^* is the same as the restriction of $\bar{\xi}^*$ to the volume $[-m, m]$. Thus, $y=\bar{x}|_{[-m,\, m]}$. Note that the terms containing J^uf_i are all canceled out since $(S^i\bar{x})_0=y_i$. We have,

$$D=\left|\sum_{i,\, j=-m,\, -n}^{m,\, n} \varphi_0(h(F^jS^i\bar{x}))-\sum_{j=-n}^{n} -\log J^u\Phi_m(h_m(F_m^j y))\right| \tag{7}$$

$$=\left|\sum_{i,\, j=-m,\, -n}^{m,\, n} (w_{mi}(F^jy)-\psi(F^jS^i\bar{x}))\right| \tag{8}$$

By the estimation (1) we have

$$|w_{mi}(F^jy)-\psi(F^jS^i\bar{x})|\leqslant \varepsilon_0 e^{-\beta d(i,\, m)}$$

where $\beta>0$ and $d(i, m)=\min\{m-i, i+m\}$.

Thus, we have

$$D=\left|\sum_{i,\, j=-m,\, -n}^{m,\, n} (w_{mi}(F^jy)-\psi(F^jS^ix))\right|$$
$$\leqslant (2n+1)\sum_{i=-m}^{m} \varepsilon_0 e^{-\beta d(i,\, m)}\leqslant C(2n+1)$$

where $C>0$ is a constant.

Therefore, we have

$$\left|b_{mn}-\frac{1}{2m}a_{mn}\right|\leqslant \frac{1}{4mn}|\log(e^{C(2n+1)})|=\frac{C}{2m}\cdot\frac{2n(m)+1}{2n(m)}$$

which goes to zero as $m\to\infty$.

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